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The Analytical Solution of Charged Black Hole and Calculation of Quantities (T, s, f)

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Abstract One of the most remarkable features of black hole is the connection between properties of the classical solutions and thermodynamics. We include the electric and magnetic charges and this lead us to resolve Einstein equations. We obtain thermodynamic properties, such as temperature, entropy density and speed of sound with analytical solution. In that case we characterize equation of state in to $V(\phi)$ language.

Keywords Black hole · Thermodynamic · Temperature

1 Introduction

Black holes remain one of the most fascinating and intriguing objects in general relativity and hide behind their horizon a singularity. The area surrounding this singularity is a region of extremely strong gravity [1]. Wheeler anticipated that "collapse leads to a black hole endowed with mass and charge and angular momentum, no other free parameters" [2]. He stressed that quantum number such as baryon number or strangeness can have no place in the external description of a black hole [3–6]. Also he focused on mass, electric charge and angular momentum because they are all conserved quantities subject to a Gauss law. We note that the magnetic charge also is conserved in Einstein-Maxwell theory [7, 8].

In the other hand a number of classical theorems show that stationary, asymptotically flat, vacuum black hole are completely characterized by its mass and spin [9–11] and their horizons have spherical topology [12]. As we know with the discovery of Hawking radiation [13] the black holes are endowed with thermodynamic properties such as entropy and temperature. In higher dimensional space times with compactifying extra dimension, the situation can not be so simple, this work was done in detail by the authors in Ref. [14].

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All these give motivation to study this subject, so in this paper we extend the analysis of [14] and include the electric and magnetic charges. In that case we want to know what In happen the thermodynamic properties. The answering of these question is the main purpose of the present paper.

2 The Equation of States

In order to study the properties of black hole, we consider the following action [15],

$$S = \frac{1}{2k_5^2} \int d^5x\sqrt{g}\left(R - \frac{1}{2}(\partial\phi)^2 - \frac{f(\phi)}{4}F_{\alpha\beta}^2 - V(\phi)\right), \tag{1}$$

where

$$f(\phi) = \frac{1}{1+\ell^2\phi^2},$$

and the corresponding metric to the action will be following ansatz,

$$ds^2 = e^{2A}(-hdt^2 + d\vec{x}^2) + e^{2B}\frac{dr^2}{h}, \tag{2}$$

where A, B, h and ϕ are function of r.

When h has a simple zero, we have a regular horizon, in this case, we assumed that A and B are finite and regular at $r = r_H$.

The properties of black hole such as temperature and entropy density are produced by the following formulas,

$$T = \frac{e^{A(r_H)-B(r_H)}}{4\pi}|h'(r_H)|, \tag{3}$$

$$S = \frac{2\pi}{k_5^2}e^{3A(r_H)}. \tag{4}$$

And also the effective number of degrees of freedom available to a system is:

$$f = -\frac{2}{3}\log\frac{s}{T^3}. \tag{5}$$

In order to compute (3), (4) and (5) we have to resolve the Einstein equation and find A, B and h.

By choosing the suitable gauge as $r = \phi$, the ansatz (2) becomes,

$$ds^2 = e^{2A}(-hdt^2 + d\vec{x}^2) + e^{2B}\frac{d\phi^2}{h}, \tag{6}$$

so we have,

$$g_{\mu\nu} = \begin{pmatrix} -he^{2A} & 0 & 0 & 0 & 0 \\ 0 & e^{2A} & 0 & 0 & 0 \\ 0 & 0 & e^{2A} & 0 & 0 \\ 0 & 0 & 0 & e^{2A} & 0 \\ 0 & 0 & 0 & 0 & \frac{e^{2B}}{h} \end{pmatrix}, \tag{7}$$

where,

$$g_{00} = -he^{2A}, \qquad g_{11} = g_{22} = g_{33} = e^{2A}, \qquad g_{44} = \frac{e^{2B}}{h}.$$

From the following Einstein equation and energy momentum tensor,

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu}, \tag{8}$$

$$\begin{gathered} G_{\mu\nu} = \frac{8\pi}{c} G_N T_{\mu\nu} = \frac{8\pi}{c} \frac{\kappa_5^2}{8\pi} T_{\mu\nu} = \kappa_5^2 T_{\mu\nu}, \\ T_{\mu\nu} = \partial_\mu \phi \partial_\nu \phi + \mathcal{L} g_{\mu\nu} \end{gathered} \tag{9}$$

one can obtain Ricci scalar as follows,

$$R = e^{-2B}(-h'' - 9h'A' + B'h' - 8hA'' + 8hA'B' - 20hA'^2). \tag{10}$$

The $\phi\phi$ part of Einstein equation will be as,

$$6h'A' + h(24A'^2 - 1) + 2e^{2B}\left[V(\phi) + \frac{f(\phi)}{2}(E^2 + B_M^2)\right] = 0, \tag{11}$$

where primes denote $d/d\phi$.

A combination of the *tt* and *xx* Einstein equation allows to find following equation,

$$h'(4A' - B') + h'' = 0. \tag{12}$$

By using (11), (12) and scalar equation of motion we obtain following expression,

$$\frac{1}{6} - A'B' + A'' = 0, \tag{13}$$

and

$$4A' - B' + \frac{h'}{h} - \frac{e^{2B}}{h} V'(\phi) - \frac{e^{2B}}{2h} f'(\phi)(E^2 + B_M^2) = 0. \tag{14}$$

In order to resolve those equation, we specify a generating function, where $A'(\phi) = G(\phi)$ then we will arrive at

$$\begin{gathered} \frac{dA(\phi)}{d\phi} = G(\phi) \quad \longrightarrow \quad \int_{A_0}^{A(\phi)} dA = \int_{\phi_0}^{\phi} G(\tilde{\phi}) d\tilde{\phi}, \\ A(\phi) = A_0 + \int_{\phi_0}^{\phi} d\tilde{\phi} G(\tilde{\phi}). \end{gathered} \tag{15}$$

By using of (13) and (15) we obtain following equations,

$$G' - GB' + \frac{1}{6} = 0,$$

and

$$B' = \frac{\frac{1}{6} + G'(\phi)}{G(\phi)} \quad \longrightarrow \quad \int_{B_0}^{B(\phi)} dB = \int_{\phi_0}^{\phi} \frac{\frac{1}{6} + G'(\tilde{\phi})}{G(\tilde{\phi})} d\tilde{\phi},$$
$$B(\phi) = B_0 + \int_{\phi_0}^{\phi} \frac{\frac{1}{6} + G'(\tilde{\phi})}{G(\tilde{\phi})} d\tilde{\phi}. \tag{16}$$

In order to compute $h(\phi)$, we continue similar way and consider (12),

$$\int_{\phi_0}^{\phi} \frac{h''}{h'} d\phi = \int_{\phi_0}^{\phi} (B' - 4A') d\tilde{\phi} \quad \longrightarrow \quad h'(\phi) = h'(\phi_0) e^{4A_0 - B_0} e^{B(\phi) - 4A(\phi)} = h_1 e^{B(\phi) - 4A(\phi)},$$
$$h(\phi) = h_0 + h_1 \int_{\phi_0}^{\phi} e^{B(\tilde{\phi}) - 4A(\tilde{\phi})} d\tilde{\phi}. \tag{17}$$

Now we solve $V(\phi)$ in (11), so we have,

$$V(\phi) = \frac{he^{-2B}}{2}\left[1 - 24G^2 - 6G\frac{h'}{h} - e^{2B} f(\phi)(E^2 + B_M^2)\right]. \tag{18}$$

Evaluating (11) and (14) at horizon one can obtain,

$$h(\phi_H) = 0 \quad \longrightarrow \quad 6h'A'G(\phi_H) = -2e^{2B}\left[V(\phi_H) + \frac{f(\phi_H)}{2}(E^2 + B_M^2)\right],$$
$$V(\phi_H) = -3e^{-2B(\phi_H)} G(\phi_H) h'(\phi_H) - \frac{f(\phi_H)}{2}(E^2 + B_M^2), \tag{19}$$

$$4G(\phi_H)h(\phi_H) + h'(\phi_H) - B'(\phi_H)h(\phi_H) - e^{2B(\phi_H)} V'(\phi_H)$$
$$- \frac{e^{2B(\phi_H)}}{2} f'(\phi_H)(E^2 + B_M^2) = 0, \tag{20}$$
$$V'(\phi_H) = h'(\phi_H) e^{-2B(\phi_H)} - \frac{f'(\phi_H)}{2}(E^2 + B_M^2).$$

And now we try to develop a power series solution around the horizon,

$$V(\phi_H) = -3G(\phi_H)\left[V'(\phi_H) + \frac{f'(\phi_H)}{2}(E^2 + B_M^2)\right] - \frac{f(\phi_H)}{2}(E^2 + B_M^2), \tag{21}$$

and the $G(\phi)$ will be following,

$$G(\phi_H) = -\frac{1}{3} \frac{[V(\phi_H) + \frac{f(\phi_H)}{2}(E^2 + B_M^2)]}{[V'(\phi_H) + \frac{f'(\phi_H)}{2}(E^2 + B_M^2)]}. \tag{22}$$

By considering $U(\phi_H) = V(\phi_H) + \frac{f(\phi_H)}{2}(E^2 + B_M^2)$, we have,

$$G(\phi) = -\frac{1}{3}(\frac{U(\phi_H)}{U'(\phi_H)} + \frac{1}{3}\left(\frac{U(\phi_H)U''(\phi_H)}{U'(\phi_H)^2} - 1\right)(\phi - \phi_H) + O[(\phi - \phi_H)^2] + \cdots. \tag{23}$$

To realize the asymptotic behavior distance of the horizon, we consider following case, where $V(\phi)$ has a maximum at $\phi=0$ with $V(0)<0$ and $V''(\phi)<0$,

$$V(\phi)=-\frac{6}{L^2}+\frac{1}{2}m^2\phi^2,\quad m^2<0, \tag{24}$$

thus ϕ is a tachyon whose mass determines the dimension of a dual operator $\mathcal{O}_\phi$ in the field theory. In a more standard gauge, we set $B=0$ instead of $r=\phi$ so in this case we have,

$$ds^2=e^{2A}(-hdt^2+d\vec{x}^2)+\frac{dr^2}{h}. \tag{25}$$

Asymptotically as $r\longrightarrow\infty$, one requires $A(r)\longrightarrow r/L$, $h(r)\longrightarrow 1$, and $\phi\approx e^{(\Delta-4)A}$ [14, 15], so we have,

$$\begin{aligned}\log\phi&=A(\Delta-4),\\ A(\phi)&=\frac{\log\phi}{\Delta-4}.\end{aligned} \tag{26}$$

We must found $A(\phi)$ to compute the temperature and entropy density for this aim we compare (17) and (26) to each other, and set: $A_H=A(\phi_H)$,

$$\begin{aligned}A(\phi)&=A_H+\int_{\phi_H}^{\phi}G(\tilde{\phi})d\tilde{\phi}=\frac{\log\phi}{\Delta-4},\\ A_H&=\frac{\log\phi}{\Delta-4}-\int_{\phi_H}^{\phi}G(\tilde{\phi})d\tilde{\phi}=\frac{\log\phi}{\Delta-4}+\int_0^{\phi_H}G(\tilde{\phi})d\tilde{\phi}-\int_0^{\phi}G(\tilde{\phi})d\tilde{\phi},\end{aligned} \tag{27}$$

and

$$A_H=\frac{1}{\Delta-4}\left(\int_{\phi_H}^{\phi}\frac{d\tilde{\phi}}{\tilde{\phi}}+\log\phi_H\right)+\int_0^{\phi_H}G(\tilde{\phi})d\tilde{\phi}-\int_0^{\phi}G(\tilde{\phi})d\tilde{\phi},$$

by taking $\phi\longrightarrow 0$ we have:

$$A_H=\frac{\log\phi}{\Delta-4}+\int_0^{\phi_H}\left[G(\phi)-\frac{1}{(\Delta-4)\phi}\right]d\phi, \tag{28}$$

finally we can obtain entropy density as a following,

$$s=\frac{2\pi}{\kappa_5^2}e^{3A(\phi_H)}=\frac{2\pi}{\kappa_5^2}\phi_H^{\frac{3}{(\Delta-4)}}\exp 3\int_0^{\phi_H}\left[G(\phi)-\frac{1}{(\Delta-4)\phi}\right]d\phi. \tag{29}$$

By comparing (6) and (25) to each other we have,

$$\frac{dr^2}{h}=e^{2B}\frac{d\phi^2}{h}\quad\longrightarrow\quad\frac{dr}{d\phi}=-e^B, \tag{30}$$

the sign is based on assuming that ϕ increases from 0 to positive values as r decreases from ∞ to finite values. From the relation (25) ($dA/dr\longrightarrow 1/L$) and (30) we arrive the following expression,

$$G=\frac{dA}{d\phi}=\frac{dr}{d\phi}\frac{dA}{dr}\approx -e^B\frac{1}{L}, \tag{31}$$

where

$$1 \approx -LG(\phi)e^{-B(\phi)}.$$

Combining this results with (3) gives us following expressions,

$$T = \frac{e^{A_H - B_H}}{4\pi}|h'(\phi_H)| \approx \frac{e^{A_H - B_H}}{4\pi}h'(\phi_H)LG(\phi)e^{-B(\phi)},$$

$$T = \frac{e^{A_H - 2B_H}}{4\pi}Lh'e^{B(\phi_H)}e^{-\log\frac{G(\phi_H)}{G(\phi)}}G(\phi_H)e^{-B(\phi)},$$

$$T = \frac{LG(\phi_H)}{4\pi}h'(\phi_H)e^{-2B(\phi_H)}e^{(A_H + B(\phi_H) - \log\frac{G(\phi_H)}{G(\phi)})}e^{-B(\phi)}.$$

And also (16) lead us the following equation,

$$B(\phi_H) - B(\phi) = -\int_{\phi_H}^{\phi}\frac{G'(\phi)}{G(\phi)}d\phi - \int_{\phi_H}^{\phi}\frac{1}{6}\frac{d\phi}{G(\phi)},$$

when $\phi \longrightarrow 0$, we have

$$B(\phi_H) - B(\phi) = \log\frac{G(\phi_H)}{G(\phi)} + \int_0^{\phi_H}\frac{1}{6}\frac{d\phi}{G(\phi)},$$

$$T = \frac{LG(\phi_H)}{4\pi}h'(\phi_H)e^{\int_0^{\phi_H}\frac{1}{6}\frac{d\phi}{G(\phi)}}\phi_H^{\frac{1}{(\Delta-4)}}e^{\int_0^{\phi_H}(G(\phi) - \frac{1}{(\Delta-4)\phi})}e^{-2B_H}.$$

By using (19) and $V(0) = -6/L^2$, we find temperature as follow,

$$T = \frac{\phi_H^{\frac{1}{(\Delta-4)}}}{2\pi L}\frac{[V(\phi_H) + \frac{f(\phi_H)}{2}](E^2 + B_M^2)}{V(0)}\exp\int_0^{\phi_H}\left[G(\phi) - \frac{1}{(\Delta-4)\phi} + \frac{1}{6G(\phi)}\right]d\phi. \tag{32}$$

The effective number of degrees of freedom available to a system is,

$$\frac{s}{T^3} = \frac{(2\pi)^4 L^3}{\kappa_5^2}\frac{V(0)^3}{[V(\phi_H) + \frac{f(\phi_H)}{2}(E^2 + B_M^2)]^3}\exp\left[-3\int_0^{\phi_h}\frac{d\phi}{6G(\phi)}\right]. \tag{33}$$

And from (5) one can obtain f,

$$f = -\frac{2}{3}\log\frac{s}{T^3} = -2\left(\log\frac{(2\pi)^{\frac{4}{3}}LV(0)}{\kappa_5^{\frac{2}{3}}[V(\phi_H) + \frac{f(\phi_H)}{2}(E^2 + B_M^2)]}\right) + 2\int_0^{\phi_H}\frac{d\phi}{6G(\phi)}. \tag{34}$$

The specific heat (heat capacity per unit volume) c_V at constant volume and density is determined by,

$$c_V = T\left(\frac{\partial s(T)}{\partial T}\right)_\rho = \frac{6\pi}{\kappa_5^2}\left(\frac{TV(0)2\pi Le^{-\int_0^{\phi_H}\frac{d\phi}{6G(\phi)}}}{[V(\phi_H) + \frac{f(\phi_H)}{2}(E^2 + B_M^2)]}\right)^3. \tag{35}$$

3 Conclusion

In this paper, we explore various aspects of magnetically charged black hole. To summarize, we have found a precise way to describe solutions for this type of black holes in five dimension. We defined suitable an ansatz and resolved the Einstein equation. We further more explained how our ansatz can be used to study the thermodynamics of magnetically charged black hole. Our calculation gives actual corrections for thermodynamic properties. We do not attain a completed characterization of the classical geometry of black holes in this paper, but hope that our results prove useful for future efforts in this direction. Obviously, it would be very interesting to generalize our construction to other more complicated geometries.

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